Math 206A Lecture 22 Notes

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1 Weak Steinitz Theorem and Robbins' Conjecture

1.1 Weak Steinitz theorem

This is the last ingredient we need to prove Gluck's theorem.

Lemma 1.1 (weak Steinitz theorem)**.** *Let $G = (V, E)$ be a plane triangulation.*[1] *Then there exists a polytope $P \subseteq \mathbb{R}^3$ with graph $G(P) \simeq G$.*

You might think this is obvious, since you can just pretend the graph is you looking at the polytope from above. But there are actually counterexamples to that approach.

Proof. Proceed by induction. The basecase is $G = K_4$, which si a square pyramid. There exists a vertex in V with $\deg(v) \leq 5$. We have 3 cases:

1. $\deg(v) = 3$: If we remove v, we still have a triangulation. Now take the polytope with this graph, and add a small pyramid to a face.

2. $\deg(v) = 4$: If we remove v, add an edge to the resulting quadrilateral, take the polytope with thi graph, and take a vertex in the middle of the added edge. Raise it up ε and add edges to the remaining 2 vertices on the boundary of the face.

3. $\deg(v) = 5$: This case is more difficult than the other cases. It involves transforming the polytope using an affine transformation to get it to look nice. $\square$

Here is a conjecture.[2]

Theorem 1.1. *For every triangulation $G = (V, E)$ with n vertices. There exists a convex polytope $P \subseteq \mathbb{R}^3$ with graph G and integer coordinates $\leq n^{10000}$.*

It is known that this is $\leq c^n$ for some constant c.

[1] We can assume straight edges, since every planar triangulation can be written with straight edges.

[2] Professor Pak came up with the above proof to try to resolve this conjecture, but the method doesn't actually work. The issue is in the $n = 5$ case.

1.2 Robbins' conjecture

Theorem 1.2 (Robbins' conjecture[3])**.** *Let* $A = A(a_1, \ldots, a_N)$ *be the area of the inscribed convex polygon with sides* $a_1, \ldots, a_n$*. Then*

1. *There exists a polynomial* $f_n(x) = c_0 x^N + c_1 x^{N-1} + \cdots + c_N$ *such that* $c_i \in \mathbb{Z}[a_1^2, \ldots, a_n^2]$ *such that* $f_n(A^2) = 0$*.*
2. *If* $n = 2k+1$*,* $N(n) = \frac{2k+1}{2}\binom{2k}{k} - e^{2k-1}$*. If* $n = 2k+2$*, then* $N(n) = 2N(n-1)$*.*

Example 1.1. If $n = 4$, then $A^2 = (\rho - a)(\rho - b)(\rho - c)(\rho - d)$, where $\rho = (a+b+c+d)/2$.

[3] Robbins figured out the first part, but he was diagnosed with a terminal illness and put out a request for someone to prove the second part before he died. Professor Pak and a student proved the second part, but when they tried to contact Robbins, they found that he had passed away a week prior.